M-theory Compactification and Two-Brane/Five-Brane Duality.

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Abstract

We discuss various dual *pairs* of M-theory compactifications. Each pair consists of a compactification in which the two-brane plays the central role in relation to a string theory and a compactification in which the five-brane takes center stage. We show that in many examples such dual pairs are interchanged by the *same* duality transformation in each case.

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1 Introduction.

Orbifolds of M-theory have recieved a substantial amount of attention in recent months [1-8]. These studies have proved crucial in formulating links between the various string theories and M-theory itself. This paper is devoted to studying further orbifolds of the eleven dimensional theory, connecting these orbifolds to string theories and gaining a small insight into the properties of M-theory itself.

Specifically, we will be interested in the relationship between the two-brane and five-brane of M-theory and the roles they play in compactifications which are related to string theories in an appropriate limit. Evidence is mounting that there exists a duality between different compactifications of M-theory, which not only acts on the compactification space, but exchanges two-branes with five-branes [9].

One example of this is M-theory on $S^1/Z_2 \times T^3$ and $K3$. Both of these are believed to be equivalent to the heterotic string on T^3 [1, 10]. In fact, the strong coupling limit of the heterotic string in seven dimensions is described by *both* of these M-theory compactifications (in the former case with a long line segment, and with a $K3$ of large volume in the latter). In other words, we have two eleven dimensional descriptions of the *same* physics. It therefore seems possible that there exists a duality between these compactifications of M-theory; after all, if our experience with string theory is anything to go by, two compactifications which describe the same physics are usually related by a duality transformation. We will refer to M-theory compactifications which describe the *same* string theories in an appropriate limit, as dual compactifications. We will later give evidence for a duality transformation which maps between such dual theories.

In M-theory on $S^1/Z_2 \times T^3$ it is the two-brane wrapped around the line segment which is identified with the fundamental heterotic string, with a coupling constant proportional to a power of the segment length [1]. In the dual description ie on $K3$, the heterotic string may be identified with a five-brane wrapped around $K3$, with heterotic coupling proportional to a power of the $K3$ volume [10, 12]. If we make the assumption that there is a duality transformation which interchanges these two compactifications, then this tranformation appears to exchange two-branes with five-branes.

We will be interested in giving further evidence for such a duality. We will give some more examples of the situation in which one finds *two* different

M-theory compactifications which are related to the *same* (or more generally, physically equivalent) string theory compactification. In one of these pairs of M-theory compactifications the two brane plays the central role, with the five-brane taking center stage in the dual compactification. In fact, we will show that in the three dual pairs of M-theory compactifications that we will discuss explicitly (and also a much larger class which were discussed in [5], as well as several of the models considered in [17, 18]), the transformation which maps between the dual M-theory compactifications, is in fact the *same* transformation. This makes the proposal for such a duality much more appealing.

The strategy for this will be to find two different compactifications of M-theory which are related to the *same* string theory compactification. As we will see, the particular string compactifications we discuss have some interesting subtleties which have a very natural interpretation in *both* M-theory compactifications, which is compelling evidence for the proposed equivalences.

There are two properties of M-theory which will play a crucial role:
(i): The two-brane and five-brane wrapped around S^1 and T^4 respectively are both equivalent to an elementary Type IIA string (see [11] and refs. therein.). Similarly, the two-brane and the five-brane wrapped around S^1/Z_2 and $K3$ respectively are both equivalent to an elementary heterotic string [1, 12].
(ii): In certain cases [2], the "twisted sector" of orbifold compactifications of the theory consists of two-branes [6] and five-branes [2].

Finally, because this work is closely related to recent work of Sen [6, 7], we will follow the notations introduced in [6, 7]. These are the following:
I_n will denote the Z_2 isometry of an $n-torus$ which reflects all n coordinates[2].
F_L will denote the left moving fermion number operator in the Type IIA or Type IIB string theories. Ω denotes the world-sheet parity operator in the Type IIB theory. Finally, by abuse of notation, we will denote orbifold isometry groups by their generators, eg instead of writing T^n/Z_2, we will write T^n/I_n.

In the next section, we will briefly review two orbifold compactifications of M-theory which will play a central role in what follows. Then in sections three and four we will give some examples of dual M-theory compactifi-

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[2]When n is odd we will specifically be discussing M-theory or Type IIA compactifications. In these cases the transformation I_n must be combined with a transformation which changes the sign of the three-form potentials in both these theories. This will be understood in what follows.

cations to two dimensions. Some of the string theory analogues of these compactifications were studied recently in [7]. In section five we show that many M-theory compactifications have a dual description, with the role of the five-brane being interchanged with the two-brane. In all the cases we have considered, these dual compactifications are mapped into one another by the *same* duality transformation.

2 M-theory on Two Orbifolds.

In this section we will briefly review the relevant features of M-theory compactified on T^n/I_n for $n = 5$ and $n = 8$. These orbifolds were discussed in [2, 3, 6] and this section is intended as a brief review of some relevant features contained in those references.

(i): T^5/I_5

The massless bosonic fields of M-theory are the metric and three-form potential. On T^5/I_5, the untwisted sector massless spectrum ie the massless states of M-theory on T^5 which survive the I_5 projection, consist of the six dimensional chiral $N = 2$ supergravity multiplet plus five tensor multiplets (which contain anti self-dual two-forms). Anomaly cancellation requires a further sixteen tensor multiplets from the "twisted sector" of the theory. It was realised by Witten [2] that these multiplets arise naturally as the world volume fields of sixteen M-theory fivebranes located on the internal space. Compelling evidence was given in [2], that this compactification of M-theory, for a particular configuration of the five-branes, is equivalent to Type IIB string theory on $K3$, with a coupling constant proportional to the radius of *any* of the circles of T^5/I_5. In particular, this leads to the conjecture that M-theory on $T^5/I_5 \times X$ is equivalent, on shrinking of the first factor, to Type IIB string theory on $K3 \times X$, with X being any space. In this paper we will be interested in the cases in which X is T^4 or $K3$.

(ii) M-theory on T^8/I_8.

This M-theory compactification was discussed in [6]. It was realised in [6] that the transformation I_8 gets mapped to a particular transformation in the Type IIA theory on T^7. This is $(-1)^{F_L}.I_7$. However, this tranformation of the IIA theory on T^7 is equivalent to the world-sheet parity transformation of the Type IIB theory and the map between these two theories involves inverting the radii of all seven circles. Thus, it was conjectured in [6] that M-theory

on T^8/I_8 is equivalent to Type IIB theory on T^7/Ω. This is by definition the Type I theory on T^7, and hence is equivalent by Type I- heterotic duality [13] to the heterotic string theory on T^7. When viewed as an orbifold of Type IIB theory, the Type I theory has a "twisted sector" spectrum consisting of 32 nine-branes [14]. Now, retracing these nine-branes back to the Type IIA theory involves inverting the radii of all seven circles of T^7 thus converting these nine-branes into 32 two-branes. However, from the Type IIA point of view, of these 32 two-branes, only 16 are independent, due to the action of the orbifold group. Finally, since a two-brane of the Type IIA theory is also a two-brane of M-theory, we see that the "twisted sector" spectrum of M-theory on T^8/I_8 is sixteen two-branes moving on the internal orbifold. Each of these two-branes carries one vector multiplet of the $N = 8$ supersymmetry of the three dimensionsal theory [6].

We will later be interested in a further reduction of this M-theory compactification on S^1 and S^1/I_1.

3 Examples of Dual M-theory Compactifications.

(i): M-theory on: $(A) = T^5/I_5 \times T^4$ and $(B) = T^8/I_8 \times S^1$

Consider M-theory compactified on both of these spaces. They are, respectively, toroidal compactifications to two dimensions of the two cases considered in the previous section. Consider case (A) first. Because we are discussing compactifications to two dimensions, massless scalars decompose into left and right movers. We will denote by (L, R) a two dimensional compactification with L left moving scalars and R right moving ones.

In case (A), the untwisted sector spectrum of M-theory is just the spectrum of M-theory on T^9 which is invariant under I_5. This is just the $N = (8, 8)$ supergravity multiplet with a scalar field content of $(64, 64)$ which together with their superpartners form representations of the supersymmetry. The "twisted sector" of this theory is just the twisted sector of M-theory on T^5/I_5, toroidally compactified to two dimensions. As realised by Witten [2], and as reviewed above, the "twisted sector" consists of 16 five-branes. Now, a five-brane of M-theory wrapped on T^4 is just an elementary Type IIA string. Each of these strings contributes $(8, 8)$ to our spectrum of massless

fields, giving a total of $(192, 192)$ scalars.

According to [2] the M-theory compactification we have just discussed should go over to Type IIB on $K3 \times T^4$, upon shrinking of one of five circles. It is straightforward to check that this compactification of Type IIB theory gives the correct spectrum. Further, this compactification of Type IIB is equivalent by T-duality to the Type IIA theory on $K3 \times T^{4\prime}$, which is in turn equivalent [10] to the heterotic string on an eight torus, T^8.

Very recently, it was realised by Sen [7] that the Type IIB theory on $K3 \times T^4$ is dual, by a sequence of duality transformations, to Type IIA string theory on T^8/I_8. As discussed in [15] certain compactifications of Type IIA and heterotic strings to two dimensions contain some subtleties due to the existence of one-loop tadpoles associated with the NS sector two-forms. It was argued in [7] and commented upon in [20] that the inconsistencies associated with such tadpoles can be cured by the introduction of n elementary strings, where n is an integer which characterises the "extent" of the tadpole inconsistency. Now, as we discussed above, from the M-theory point of view, we find $(64, 64)$ states in the untwisted sector together with $16 \times (8, 8)$ which are carried by elementary Type IIA strings which come from the 16 five-branes wrapped around T^4. We therefore can expect to see these elementary strings present in the Type IIA theory on T^8/I_8. In fact this orbifold of Type IIA theory was analysed by Sen [7], where he showed that precisely 16 elementary IIA strings are required to cure the one-loop tadpole inconsistency. In the untwisted sector, we find the $(64, 64)$ states which were present in the untwisted sector of the M-theory compactification. Thus, this Type IIA compactification also has the same spectrum and it is indeed compelling that the extra strings are required not only from the M-theory point of view, but for consistency of the Type IIA theory itself.

Now let us consider case (B) ie M-theory on $T^8/I_8 \times S^1$. We reviewed the case of M-theory on T^8/I_8 in the previous section. Thus all that remains is to reduce this model on S^1. In the untwisted sector, we again find the $(64, 64)$ spectrum of massless states. The twisted sector consists of sixteen two-branes wrapped on S^1, which are equivalent to sixteen Type IIA strings. However, this compactification of M-theory is equivalent to Type IIA theory on T^8/I_8, which as we saw above, also has the same spectrum.

We have thus given evidence that M-theory on $T^5/I_5 \times T^4$ is physically equivalent to Type IIB on $K3 \times T^4$, the heterotic/Type I theory on T^8, and finally the Type IIA theory on T^8/I_8. In this M-theory compactification,

the five-brane played a crucial role. Secondly, we have seen that in M-theory on $T^8/I_8 \times S^1$, two-branes play a crucial role. Further, this theory is *also* equivalent to Type IIA on T^8/I_8. This strongly suggests that there is a symmetry of M-theory in which five-branes are interchanged with two-branes. We will identify such a symmetry after the next section.

We now go on to discuss in a similar fashion two more M-theory compactifications, both of which appear to be equivalent to a *single* compactification of heterotic string theory. Again, we will see that in one case the five-brane plays a crucial role, whereas in the other case it is the two-brane which does so.

4   Further Examples.

In this section we will again consider M-theory compactified on two different spaces. These examples will be very similar to the cases considered in the previous section, except we will replace the T^4 factor in case (A) with $K3$ and the S^1 factor in case (B) with S^1/I_1. We therefore expect to see the elementary Type IIA strings being replaced with elementary heterotic strings.

(i): M-theory on $T^5/I_5 \times K3$.

This compactification should be equivalent to the Type IIB string theory on $K3 \times K3$ and also to a particular orbifold of the heterotic string in two dimensions [2].

In the M-theory case, we must simply reduce the model of [2] on $K3$. Let us first analyze this model as an orbifold of M-theory on $T^5 \times K3$. In the untwisted sector, we find the massless states in the effective two-dimensional theory comprise the chiral $N = (8, 0)$ supergravity multiplet and (192,192) scalars. The twisted sector of this model consists of the twisted sector of the model considered in [2], reduced on $K3$. As noted above, the twisted sector of the model in [2] consisted of sixteen fivebranes. The bosonic world volume field content of the fivebrane consists of an anti self-dual two-form potential and five scalars. Thus, to calculate the spectrum of our model, we simply need to wrap these sixteen fivebranes on $K3$. A fivebrane in M-theory wrapped on $K3$ is equivalent to a string with the world sheet field content of the heterotic string [12]. Thus in the notation above, the double dimensional reduction of the fivebrane on $K3$ gives rise to (24,8) scalars in the two dimensional theory. These are of course the world sheet scalar degrees

of freedom of the heterotic string in light cone gauge. Thus, as we have 16 fivebranes in six dimensions, the massless scalar content of this M-theory compactification consists of (192,192) + 16(24,8) = (576,320). Given the number of supersymmetries, it can be checked that the full spectrum is free from anomalies. We will shortly see how these sixteen elementary heterotic strings naturally arise in an orbifold of the heterotic string compactified to two dimensions.

According to [2], the above M-theory compactification should be equivalent to the Type IIB string on $K3 \times K3$. In ten dimensions, the massless bosonic field content of the Type IIB theory consists of two scalars, two 2-form potentials, a 4-form potential and the metric. The 4-form potential is self-dual. The product manifold $K3 \times K3$ has the following non-zero Betti numbers: $b_0 = 1$, $b_2 = 44$, $b_4^+ = 371$, $b_4^- = 115$, $b_6 = 44$ and $b_8 = 1$. Reducing the two 2-forms on this manifold gives rise to (88,88) scalars. Reducing the self-dual four form gives (371,115) scalars. Because it is possible to define a torsion free $Spin(7)$ structure on R^8 which is preserved by the holonomy structure of $K3 \times K3$, the moduli space of metrics on $K3 \times K3$ has dimension $b_4^- + 1 = 116$ [16]. Equivalently, each $K3$ has a moduli space of metrics of dimension 58, giving 116 for the product. However, one of these components becomes part of the supergravity multiplet, as the metric has formally -1 degrees of freedom in two dimensions. This means that the metric contributes (115,115) scalars to the 2d theory. Finally, each ten dimensional scalar decomposes into a left moving scalar and a right moving one. All in all, the field content of this theory is precisely that of the M-theory compactification.

A more difficult problem is to relate this M-theory compactification to a compactification of heterotic string theory. If we exchange the two factors in the above M-theory compactification, we have M-theory on $K3 \times T^5/I_5$. Now, M-theory on $K3 \times T^5$ should [10] be equivalent to the heterotic string on $T^3 \times T^5$. Thus, in the compactification we are discussing one would naively expect that the heterotic compactification is on $T^3 \times T^5/I_5$. However, as the heterotic string is an oriented string theory, it is not possible to formulate the theory on this background[3]. We thus expect that the heterotic dual compactification should be some other Z_2 orbifold of the heterotic string on T^8.

According to [6] the transformation I_5 in M-theory gets mapped to $(-1)^{F_L} . I_4$

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[3]These arguments were made in a similar context in [19].

in the Type IIA theory on T^8 [6] [4]. We thus expect that the M-theory compactification we are discussing is equivalent to the Type IIA theory on a Z_2 orbifold of $K3 \times T^4$, where the Z_2 element is the one just described. It is possible to check that the untwisted and twisted sector spectra are the same in the M-theory and IIA theories. This compactification of Type IIA theory can now be mapped to a heterotic compactification.

On $K3$, the Type IIA theory is believed to be equivalent to the heterotic, string on T^4, which has Narain lattice, $\Gamma^{20,4}$. The transformation, $(-1)^{F_L}$ maps to the Z_2 transformation which inverts $\Gamma^{20,4}$ in the heterotic description [17]. The Type IIA compactification which gives the same spectrum as the M-theory background which interests us is mapped in the heterotic theory to a Z_2 orbifold of $\Gamma^{24,8}$, which describes the heterotic string in two dimensions. The transformation $(-1)^{F_L}$ in the Type IIA theory then corresponds to inversion of a signature $(20,4)$ factor in this lattice. The transformation which inverts all the coordinates of the T^4 factor in the Type IIA compactification is mapped to a reflection of the remaining signature $(4,4)$ piece of the lattice, since this T^4 is "common" to both the Type IIA and heterotic compactifications. This leads us to the final statement that the Z_2 transformation which defined our original orbifold in M-theory gets mapped to the following transformation in the heterotic string, toroidally compactified to two dimensions:

$$g : \Gamma^{24,8} = -\Gamma^{24,8} \tag{1}$$

ie this transformation inverts the entire lattice of the compactified string theory. We should therefore, perhaps naively, expect that our M-theory compactification is equivalent to a Z_2 orbifold of this heterotic compactification with orbifold group generated by g. In particular, this orbifold must satisfy several criteria: (i) it must be modular invariant. (ii) it should give the correct supersymmetry algebra in the effective two dimensional theory, namely $N = (8,0)$. (iii) it should give the same field content as the above theories.

It is straightforward to check that points (i) and (ii) are satisfied. To verify point (iii) additional subtleties are involved. Firstly, in the untwisted sector, the massless spectrum of the heterotic orbifold is precisely that which we found above in the M-theory case. One would therefore expect that the

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[4]The following derivation of the heterotic dual of the Type IIA model was also given in [7].

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twisted sector should contain the requisite states that we found in M-theory. However, because the left moving twisted sector ground state is massive, we will not find any massless states in the twisted sector - at least none which arise in the standard fashion as tensor products of left and right movers on the twisted worldsheet. However, the untwisted spectrum that we have in this model is the *same* as that of M-theory on $K3 \times T^5/I_5$, namely $(192, 192)$ scalars. Further, because we saw that in the M-theory case, the twisted sector consisted of 16 elementary heterotic strings, it is natural to expect that they are also present here. In the heterotic compactification we are discussing, it is therefore again natural to expect that the presence of these strings cures any inconsistencies in the one-loop tadpoles [15] that may be present. In fact, this particular orbifold of the heterotic string was also analysed by Sen [7], and it was found that precisely sixteen elementary heterotic strings are required to cancel any inconsistencies. This is of course compelling evidence that the theories are equivalent. We will now consider another M-theory compactification which is equivalent to the heterotic orbifold which we have just discussed.

(ii): M-theory on $T^8/g' \times S^1/I_1$.

Our aim is to construct an orbifold of M-theory which is equivalent to the heterotic orbifold we have just considered. Because the $E_8 \times E_8$ heterotic string in ten dimensions is believed to be equivalent to M-theory on S^1/I_1 we expect that symmetries of the heterotic string be present in this M-theory compactification. Thus, since our goal is to find another M-theory compactification which is equivalent to the heterotic orbifold of the previous section, it should be possible to *define* the required orbifold of M-theory *a priori*. This is precisely the strategy we will take.

Since the heterotic orbifold which interests us is an orbifold of the toroidally compactified heterotic string in two-dimensions, we must first consider M-theory on S^1/I_1 toroidally compactified to two dimensions. Such a compactification of M-theory contains two ten dimensional E_8 Super-Yang-Mills multiplets toroidally compactified to two dimensions. The full scalar field content of the compactification is $(192, 192)$. Of these, $(128, 128)$ originate from the ten-dimensional vector multiplets. In the heterotic orbifold which we wish to "duplicate" in M-theory, the action on T^8 is simply I_8. Thus, we expect g' to contain such a factor. Further, as we reviewed above, the "twisted sector" associated with I_8 is sixteen two-branes, which when reduced on S^1/I_1 are precisely the sixteen heterotic strings which were required for consistency of

the heterotic compactification. It therefore becomes plausible that g', which defines the M-theoretic analogue of g, is simply I_8. On $T^8/I_8 \times S^1/I_1$, M-theory contains $(64, 64)$ massless scalars which originate from the metric and three-form. However there exists the possibility that the modes associated with the ten-dimensional vector multiplets will contribute. As we mentioned above, before the I_8 projection there are precisely $(128, 128)$ such states. By symmetry, because I_8 acts identically on all $1 - cycles$ of T^8, we expect that either these $(128, 128)$ modes are odd under I_8 or they are all even. In the case when these states are odd, one can check that the corresponding heterotic string orbifold does *not* give a spectrum which matches the M-theory one. In the case when the $(128, 128)$ states are even, I_8 has precisely the same action in M-theory as g has on the heterotic string and we find complete agreement between the spectrum of states in the two compactifications. To recap, this is namely $(192, 192)$ from the untwisted sector plus $16 \times (24, 8)$ which are carried by elementary heterotic strings. From the M-theory point of view these strings come from two-branes wrapped around the line segment. Thus, we expect that the M-theoretic analogue of g is I_8.

We have thus related the heterotic string on the orbifold defined by g to M-theory on $K3 \times T^5/I_5$ and M-theory on $T^8/I_8 \times S^1/I_1$.

In the first of these cases, we saw that the elementary heterotic string may be identified as a five-brane of M-theory wrapped around $K3$ and the 16 additional heterotic strings required for consistency of the string theory have an M-theoretic interpretation as the 16 five-branes of [2] wrapped on $K3$.

In the second of these cases the elementary heterotic string is naturally identified with a two-brane of M-theory wrapped around S^1/I_1 with the additional 16 requisite heterotic strings being interpreted as the 16 two-branes of [6] wrapped around S^1/I_1.

5 Duality Symmetry.

In this concluding section we will identify the Z_2 transformation which defines a duality map between an M-theory compactification in which the two-brane plays the crucial role in the relation to a string theory and a compactification in which the five brane does so. The strategy we will follow will be to identify the required transformation in the Type IIA theory which will then allow us

to induce the requisite transformation in M-theory. We will find that the *same* transformation defines the map between all the pairs of M-theory compactifications which we have discussed in this paper as well as a much wider class which were discussed in [5] and also several of the models considered in [17, 18]. We take this as further evidence that M-theory possesses such a duality symmetry.

In [6], strong evidence was presented that for M-theory on $X \times S^1$, with X any space, the transformation I_1, *if* combined with a transformation α which acts on X, is represented as $(-1)^{F_L}.\alpha$ in the Type IIA theory on X. For the case when X is a torus, T^n, we can identify the transformation I_1 in M-theory as the transformation I_1 in the Type IIA theory on $T^{n'}$ [1]. We will assume that these properties hold in general in $d < 10^5$. Thus if we consider M-theory on T^n, we can rewrite the transformation I_n as $I_1 \times I_{n-1}$. This then translates to the Type IIA theory on T^{n-1} as the transformation $(-1)^{F_L} \times I_{n-1}$. Thus, in certain cases, we may identify the transformation $(-1)^{F_L}$ in the Type IIA theory with I_1 in M-theory. Another fact that we will require is that the Type IIA theory on T^4 has a self duality transformation σ which takes $(-1)^{F_L}$ into I_4 [7]. These observations will allow us to identify the transformation which maps between the pairs of M-theory vacua that we have discussed above.

(i): M-theory on: (A): $T^5/I_5 \times T^4$ and (B): $T^8/I_8 \times S^1$

In these examples, we identified a dual Type IIA comapactification on T^8/I_8. However, upon rewriting T^8 as $T^{4'} \times T^4$ and I_8 as $I_4{}'I_4$ and using the self duality transformation σ, we can map this Type IIA theory to the Type IIA theory on $(T^{4'} \times T^4)/I_4{}'.(-1)^{F_L}$. In this example, because $(-1)^{F_L}$ is combined with another transformation, we can identify its M-theory analogue as I_1. Now, in case (A) above, we can see that the transformation $I_5 = I_1.I_4$ is represented in the Type IIA theory on $T^{4'} \times T^4$ as $(-1)^{F_L}.I_4$. σ in the Type IIA theory maps $(-1)^{F_L}$ to $I_4{}'$. Hence, $(-1)^{F_L}.I_4$ is mapped into $I_4{}'.I_4 = I_8$. Finally, I_8 in the Type IIA theory is just I_8 in M-theory, which is just case (B) above. Thus if we translate the transformation σ into its M-theory counterpart, we have a transformation which maps I_1 in case (A) to a transformation $I_4{}'$. In other words, the transformation I_5 in M-theory on T^9 is mapped to the transformation I_8, by the M-theoretic analogue of σ in the Type IIA theory. We will denote this transformation in M-theory as

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[5]In $d = 10$ it was shown in [6] that this does not hold.

σ_M for definiteness.

Using this information, we can now check if indeed σ_M has a similar action on the other pairs of examples that we considered. These were:

(ii): M-theory on (C): $T^5/I_5 \times K3$ and (D): $T^8/I_8 \times S^1/I_1$.

It is natural to restrict ourselves to the case when $K3$ is the orbifold T^4/I_4, otherwise it is not presently possible to perform the required analysis.

In case (C), the orbifold isometry group which acts on M-theory on T^9 has non-identity elements: I_5 and I_4. Under the action of σ_M these transform as follows:

$$\sigma_M : I_5 \rightarrow I_8 \tag{2}$$

$$\sigma_M : I_4 \rightarrow I_1 \tag{3}$$

Hence, the generators in case (C) are transformed to those in case (D). Thus, σ_M also defines a duality map between these two physically equivalent compactifications of M-theory.

(iii): M-theory on: (E): T^5/I_4 and (F): T^5/I_1. These two cases are S^1 compactifications of the dual pair we discussed in the introduction. Case (E) is a special case of M-theory on $K3 \times S^1$, which we expect to be equivalent to the heterotic string on T^4. Case (F) is just M-theory on $S^1/I_1 \times T^4$, which we also expect [1] to be equivalent to the heterotic string on T^4. In case (E), the transformation I_4 can be identified with the transformation $(-1)^{F_L}.I_3$ in the Type IIA theory on T^4 [6]. By radius inversion on the three circles on which I_3 acts this theory is mapped to Type IIB theory on T^4/Ω, which is just Type I theory on T^4, which is equivalent [13] to the heterotic string on T^4. σ in the Type IIA theory on T^4 maps $(-1)^{F_L}.I_3$ to $I_4 I_3 = I_1$. From the M-theory point of view, I_1 in the Type IIA theory on T^4 is also I_1 in M-theory [1]. Therefore, case (E) is mapped to case (F), by the action of σ_M.

iv: A Larger Class of Examples.

In [5] we presented evidence that M-theory compactifications on Joyce manifolds of G_2 and $Spin(7)$ holonomy are dual to compactifications of heterotic string theory on Calabi-Yau manifolds and Joyce manifolds of G_2 holonomy respectively. These theories are $N = 1$ theories in four and three dimensions respectively. In these cases, the five-brane of M-theory wrapped around a $K3$ submanifold of the Joyce manifold may be identified with the fundamental heterotic string. However, we can also expect that there exists a dual M-theory compactification in which the wrapped two-brane plays the

fundamental role, since M-theory on $S^1/I_1 \times X$ is equivalent to the heterotic string on X [1].

In fact, all the compact manifolds discussed in [5] were constructed by Joyce as blown up toroidal orbifolds. Further, the only non-freely acting orbifold generators in M-theory, were of the form I_4. Thus, by applying the transformation σ_M once, one of the generators I_4 is mapped to I_1, as we discussed in the previous case. Although we do not present the details here, it may be checked that in all the M-theory compactifications in [5] (for which the corresponding heterotic dual is a compactification on some manifold X), the transformation σ_M maps the M-theory compactification on the Joyce manifold to M-theory on $S^1/I_1 \times X$. Thus, in all the cases considered in [5] σ_M maps one M-theory compactification in which the five-brane plays the crucial role, to one in which the two-brane does so. This reasoning also applies to the cases considered in [18] and also to several of the models considered in [17].

(v): $K3$ Fibrations.

Compactifications of M-theory and string theories on manifolds which admit $K3$ fibrations have played an important role in our understanding of string theory dualities [17, 22]. In such M-theory compactifications it is the five-brane wrapped around the $K3$ fiber which one identifies as the heterotic string. If we consider a point in the moduli space in which the fiber is T^4/I_4 (assuming that such a point exists), then we can apply the transformation σ_M which exchanges the $K3$ fibers with $S^1/I_1 \times T^3$ fibers. In other words, we can apply the duality transformation σ_M fibrewise in the adiabatic limit [17]. In the dual compactification the two-brane wrapped around the line segment is identified with the heterotic string. If this argument does apply in this case, then it appears that compactifications of M-theory on manifolds which admit $K3$ fibrations also have a dual description.

6 Comments.

We have seen that, in a large number of cases, σ_M exchanges a given M-theory compactification with a dual compactification. Five-branes in one compactification are replaced with two-branes in the dual compactification, and vice-versa. It therefore appears that such a duality symmetry is a fairly general property of M-theory compactifications. However, because this dual-

ity is a property of M-theory *compactification*, it is not clear if such a duality has an eleven dimensional origin, although this does remain an open possibility. A two-brane/five-brane duality in eleven dimensions was suggested in the first reference of [9].

We hope that these results will be a small clue towards the formulation of M-theory.

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